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A simple laboratory model for the oceanic circulation

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A linear theory is developed for the motion of a viscous, incompressible fluid in a rotating cylinder with a sloping bottom.

An analysis of the normal modes of oscillation reveals that the presence of the bottom slope introduces a new set of low frequency inertial oscillations to replace the purely geostrophic modes which are not allowed in this geometry. The new waves possess mean circulation and are the mechanism by which the fluid adjusts to changes in the rotation rate of the container, a process discussed in detail.

The steady motion produced in the cylinder when the cylinder's upper surface rotates at a different rate than the bottom surface is studied. It is shown that the presence of the bottom slope inhibits the steady fluid motion in the body of the cylinder and introduces a non-symmetric, high velocity side wall boundary layer.

Experimental evidence, presented to validate the theory, reproduces certain important features of the oceanic circulation.

1. Introduction

There are two objectives of the present research. The first is to examine those nearly rigid rotating flows for which the theory of Greenspan (1965) must be modified or extended. The second is to study a simple laboratory model which is capable of reproducing certain important features of interest to the study of large-scale oceanic circulations. Both purposes are served and the ends achieved by considering a single configuration, the 'sliced' cylinder shown in figure 1. (The cylinder is bounded by the surfaces $r = r_0$, $z = L$, $z = y \tan \alpha$.)

The general theory cited above applies to any container shape whose entire surface is an envelope of *closed* geostrophic contours C, defined in the following way. If z measures distance along the rotation axis, and $z = f(x, y)$, $z = -g(x, y)$ are the equations of the top and bottom surfaces of the container, then a geostrophic contour C is a curve of constant total height $f+g$. It is important to note that the sliced cylinder possesses no closed geostrophic contours.

The analysis of the linear equations of motion, scaled with respect to the container height L, the rotation rate Ω and the deviation from rigid rotation $\epsilon\Omega L$, which in the rotating frame are

$$\partial \mathbf{q}/\partial t + 2\hat{\mathbf{k}} \times \mathbf{q} = -\nabla p + E\nabla^2 \mathbf{q}, \quad (1)$$

$$\nabla . \mathbf{q} = 0, \quad (2)$$

$$E = \nu/\Omega L^2,$$

showed that two distinct classes of inviscid natural modes exist of the form $\mathbf{Q}e^{i\lambda t}$. The geostrophic mode corresponds to zero frequency ($\lambda = 0$), and represents a flow for which the Coriolis force and pressure gradient are in exact balance.

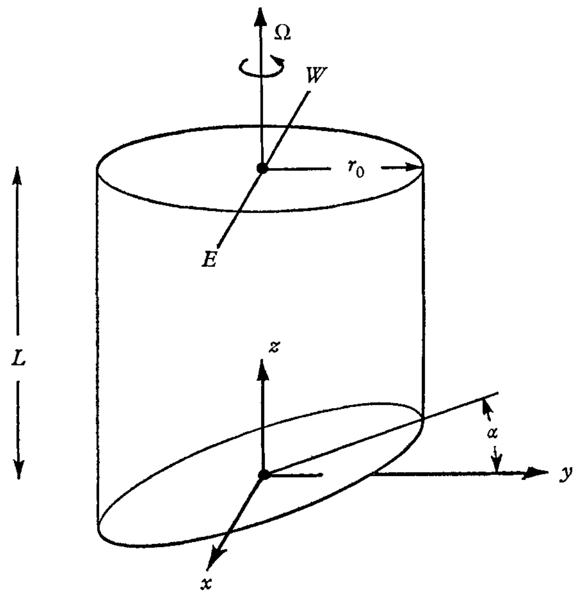

FIGURE 1. The 'sliced' cylinder configuration.

The geostrophic pressure, ϕ_0, is then a function only of total height $h = f+g$ with an extremely slow time variation

$$\phi_0 = \phi_0(h, E^{\frac{1}{2}}t).$$

The velocity is
$$\mathbf{q}_0 = -\frac{1}{2}\frac{\partial \phi_0}{\partial h}\mathbf{n}_T \times \mathbf{n}_B,$$

with
$$\mathbf{n}_T = -\nabla f + \hat{\mathbf{k}} = [1+(\nabla f)^2]^{\frac{1}{2}}\,\hat{\mathbf{n}}_T,$$
$$\mathbf{n}_B = -\nabla g - \hat{\mathbf{k}} = [1+(\nabla g)^2]^{\frac{1}{2}}\,\hat{\mathbf{n}}_B,$$

and the circulation on a geostrophic contour is in general not zero. Viscous corrections require the calculation of the $O(E^{+\frac{1}{2}})$ flow field.

Inertial modes, $\mathbf{Q}_n e^{i\lambda_n t}$; $\Phi_n e^{i\lambda_n t}$, correspond to non-zero oscillation frequencies and have a number of properties commonly associated with all eigenvalue problems. For example, $-2 < \lambda_n < 2$, and modes of different frequencies are orthogonal

$$\int \mathbf{Q}_n \cdot \mathbf{Q}_m^\dagger \, dv = 0.$$

The dagger denotes complex conjugation. Furthermore, these oscillatory modes possess no mean circulation about geostrophic contours, i.e.

$$\oint_C d\mathbf{s} \cdot \frac{1}{h}\int_{-g}^{f} \mathbf{Q}\, dz = 0,$$

and this distinction allows the complete modal synthesis of an arbitrary disturbance.

Clearly the 'sliced' cylinder configuration, for which no geostrophic mode exists because there are no closed geostrophic contours, falls outside the scope of this theory and the theory must be modified. Although the geostrophic mode is excluded in the sliced cylinder, such motion is possible (arbitrary in fact) in the right circular cylinder, $\alpha = 0$, because *any* closed curve is then a geostrophic contour. Thus if α is small (but $\alpha \gg E^{\frac{1}{2}}$) we are faced with a situation in which a slight geometrical change reduces an infinite number of modes to none. It will be shown that the geostrophic mode is then completely replaced by a new infinite set of low frequency, depth independent inertial oscillations called Rossby waves (Rossby 1939). Each of these new quasi-geostrophic inertial waves reduces to an allowable steady motion when $\alpha = 0$. The modes do possess average vorticity (the mean circulation theorem does not apply to this geometry) and with the addition of the rest of the inertial waves any initial disturbance can be represented.

2. The Rossby modes

Our concern is with the Rossby modes, all of which degenerate into the general geostrophic flow when $\alpha = 0$. To find the inviscid Rossby modes, set $E = 0$ and let

$$\mathbf{q} = \mathbf{Q}e^{i\lambda t}, \quad p = \phi e^{i\lambda t}, \tag{3}$$

with

$$\left.\begin{aligned} \mathbf{Q} &= \mathbf{Q}^{(0)} + \alpha\mathbf{Q}^{(1)} + \ldots, \\ \phi &= \phi^{(0)} + \alpha\phi^{(1)} + \ldots, \\ \lambda &= \alpha\lambda^{(1)} + \alpha^2\lambda^{(2)} + \ldots. \end{aligned}\right\} \tag{4}$$

Each of the inertial modes which exists in the case $\alpha = 0$ exists in the new configuration in a slightly altered form. Discussion of their modification is omitted. The scaling in (4) is designed for a discussion of only the Rossby wave modes.

Upon substitution of these expansions into (1) and (2), it follows that

$$\left.\begin{aligned} 2\hat{\mathbf{k}} \times \mathbf{Q}^{(0)} &= -\nabla\phi^{(0)}, \\ \nabla . \mathbf{Q}^{(0)} &= 0, \end{aligned}\right\} \tag{5}$$

with $\quad \mathbf{Q}^{(0)} . \hat{\mathbf{k}} = 0 \quad \text{on} \quad z = 0 \quad \text{and} \quad z = 1$

and $\quad \mathbf{Q}^{(0)} . \hat{\mathbf{r}} = 0 \quad \text{on} \quad r = a = r_0/L.$

The first-order problem is

$$\left.\begin{aligned} i\lambda^{(1)}\mathbf{Q}^{(0)} + 2\hat{\mathbf{k}} \times \mathbf{Q}^{(1)} &= -\nabla\phi^{(1)}, \\ \nabla . \mathbf{Q}^{(1)} &= 0, \end{aligned}\right\} \tag{6}$$

with

$$\left.\begin{aligned} \mathbf{Q}^{(1)} . \hat{\mathbf{k}} &= 0 \quad \text{on} \quad z = 1, \\ (y\,\partial\mathbf{Q}^{(0)}/\partial z + \mathbf{Q}^{(1)}) . \hat{\mathbf{k}} - \mathbf{Q}^{(0)} . \hat{\mathbf{j}} &= 0 \quad \text{on} \quad z = 0 \end{aligned}\right\} \tag{7}$$

and $\mathbf{Q}^{(1)} . \hat{\mathbf{r}} = 0$ on $r = a$.

The velocity amplitude $\mathbf{Q}^{(0)}$ can be expressed in terms of the pressure since (5) may be written as

$$\mathbf{Q}^{(0)} = \tfrac{1}{2}\hat{\mathbf{k}} \times \nabla\phi^{(0)}, \quad \partial\phi^{(0)}/\partial z = 0,$$

with $\phi^{(0)} = 0$ on $r = a$. Thus, the primary velocity field is independent of height and is geostrophic; to this order it is identical to one of the possible geostrophic modes in the right circular cylinder. The solution of the $O(\alpha)$ problem is required to determine these modes precisely.

The function $\mathbf{Q}^{(1)}$ can also be expressed in terms of $\phi^{(0)}$ by taking the curl of the momentum equations, (6), and then integrating with respect to z. The result is

$$\mathbf{Q}^{(1)} = \tfrac{1}{4}i\lambda^{(1)}(z-1)\,\nabla^2\phi^{(0)}\hat{\mathbf{k}} + \mathbf{A}(x,y), \tag{8}$$

where $\mathbf{A}$ is arbitrary. However, since $\mathbf{Q}^{(1)}.\hat{\mathbf{k}} = 0$ on $z = 1$

$$\mathbf{A}.\hat{\mathbf{k}} = 0.$$

An equation for $\phi^{(0)}$ is obtained by satisfying the boundary conditions on $\mathbf{Q}^{(1)}$ at the inclined surface, i.e. (8) is substituted into (7) yielding

$$\nabla^2\phi^{(0)} + \frac{2\hat{\mathbf{j}}}{i\lambda^{(1)}}\cdot(\hat{\mathbf{k}}\times\nabla\phi^{(0)}) = 0, \tag{9}$$

subject to the boundary condition

$$\phi^{(0)} = 0 \quad \text{on} \quad r = a.$$

The solutions of the above eigenvalue problem are

$$\left.\begin{aligned} \phi^{(0)}_{mn} &= J_m(k_{mn}r/a)\exp i[m\theta + k_{mn}(r/a)\cos\theta], \\ \lambda^{(1)}_{mn} &= a/k_{mn}, \end{aligned}\right\} \tag{10}$$

where $J_m(k_{mn}) = 0$ and n ranges over all integer values, both positive and negative. These modes have a preferred direction of phase propagation in the direction of negative x and are dynamically similar to the waves found by Rossby (1939) in his β-plane model of the atmosphere. It is well known that in certain circumstances the effect of a depth incline is dynamically similar to the β-effect and it is not surprising that the eigen modes found here are similar to those found by Longuet-Higgins (1965) and Pedlosky (1965) in their studies of two-dimensional motions in bounded ocean basins.

Obviously these Rossby waves are without a vertical structure but they do have all the properties (orthogonality, etc.) of the regular inertial oscillations. However, in distinction to the regular inertial oscillations, these modes do have an $O(1)$ mean circulation about contours of constant r. (These would be geostrophic contours if $\alpha = 0$.) The circulation is

$$\pi r(i)^m \frac{d}{dr} J_m^2\left(k_{mn}\frac{r}{a}\right).$$

The situation is entirely similar for any shaped lower surface for which closed geostrophic contours do not exist as long as the deviation from a level plane is small. If

$$z = -\alpha g(x,y)$$

is the bottom boundary, then the equation for the basic, order one pressure amplitude corresponding to (9) is

$$\nabla^2\phi^{(0)} - \frac{2}{i\lambda^{(1)}}\nabla g.(\hat{\mathbf{k}}\times\nabla\phi^{(0)}) = 0. \tag{11}$$

Although the boundary-value problem becomes more difficult, the general conclusions remain the same. If the container has no geostrophic contours, so that geostrophic modes with mean circulation are not possible, then the Rossby modes arise to compensate for the loss of a steady geostrophic flow (or part thereof). If the container has geostrophic contours, Rossby waves may still be possible, but they do not possess mean circulation. The geostrophic mode then possesses all the mean circulation and the Rossby modes have the same character as the regular inertial modes. For example, if $g = \sqrt{(x^2+y^2)}$, then circles r equal constant are geostrophic contours and Rossby wave modes are present:

$$\phi^{(0)}_{mn} \exp(i\alpha\lambda^{(1)}_{mn} t) = J_{2m}(k_{mn} r/a) \exp\{i(m\theta + \alpha\lambda^{(1)}_{mn} t)\},$$

where

$$J_{2m}(k_{mn}) = 0 \quad \text{and} \quad \lambda^{(1)}_{mn} = 8ma^2/k^2_{mn}.$$

In this case the Rossby waves propagate along the geostrophic contours and have no net circulation around a geostrophic contour unless $m = 0$, in which case $\lambda^{(1)}_{mn} = 0$ thereby degenerating the solution into one of the infinite number of purely geostrophic modes.

3. The spin-up problem

Spin-up in the sliced cylinder differs markedly from that in the symmetric configuration when $\alpha = 0$. The physical problem involves a container and fluid in an initial state of solid body rotation. The rotation rate of the container is then impulsively changed by a small amount and an adjustment to a new state of rigid rotation occurs. As viewed from a co-ordinate system moving with the container, the initial velocity distribution in the sliced cylinder is to $O(\alpha)$

$$\left.\begin{aligned} \mathbf{q}_* &= -r\hat{\boldsymbol{\theta}} = \tfrac{1}{2}\hat{\mathbf{k}} \times \nabla p_* \\ \text{or} \qquad \nabla p_* &= -2r\hat{\mathbf{r}}. \end{aligned}\right\} \tag{12}$$

For the right circular cylinder, Greenspan & Howard (1963) showed that the subsequent $O(1)$ motion is just a low viscous decay of the initial state:

$$\left.\begin{aligned} \mathbf{q} &= -r\hat{\boldsymbol{\theta}} \exp(-2E^{\frac{1}{2}}t), \\ p/a^2 &= (1-r^2/a^2)\exp(-2E^{\frac{1}{2}}t). \end{aligned}\right\} \tag{13}$$

The motion remains symmetrical and only the azimuthal velocity is affected. When the bottom surface is tilted, the ensuing motion differs dramatically from that just described. The initial state flings all fluid columns across lines of constant total height inducing vorticity relative to the rotating frame by vortex tube stretching. If $\alpha \gg E^{\frac{1}{2}}$, this vorticity change is larger than that produced by Ekman layer suction, completely altering the character of the spin-up. All the Rossby waves are excited by the initial state, and for that matter, only these depth independent inertial modes. The resulting flow exhibits a propagation of vorticity from one side of the container to the other, cells of vortical motion moving from positive to negative x.

The general solution of the problem for the primary flow variables can be represented as

$$\left.\begin{aligned} \mathbf{q} &= \Sigma A_{mn} \mathbf{Q}_{mn} \exp(i a\alpha t/k_{mn}), \\ p &= \Sigma A_{mn} \phi_{mn} \exp(i a\alpha t/k_{mn}), \end{aligned}\right\} \tag{14}$$

with $\mathbf{Q}_{mn} = \tfrac{1}{2}\hat{\mathbf{k}} \times \nabla\phi_{mn}$. (The superscript notation has been dropped.)

The orthogonality relationship is used to determine the Fourier coefficients from the initial velocity distribution

$$A_{mn} = \int \mathbf{q}_* . \mathbf{Q}^+_{mn} dV \Big/ \int \mathbf{Q}_{mn} . \mathbf{Q}^+_{mn} dV,$$

or in terms of the pressure

$$A_{mn} = \int \nabla p_* . \nabla \phi^+_{mn} dV \Big/ \int \nabla \phi_{mn} . \nabla \phi^+_{mn} dV.$$

The calculation is straightforward and the result is

$$A_{mn} = 2(i)^{-m} a^2 / k^2_{mn}. \tag{15}$$

With some algebraic manipulation it follows that

$$p/a^2 = 8 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{\delta_m}{k^2_{mn}} J_m(k_{mn} r/a) \cos[k_{mn}(r/a)\cos\theta + \alpha a t - \tfrac{1}{2} m\pi] \cos m\theta, \tag{16}$$

where

$$\delta_m = 1 \quad (m \neq 0),$$
$$\delta_m = \tfrac{1}{2} \quad (m = 0).$$

Another form convenient for computation is

$$\frac{p}{a^2} = 1 - \left(\frac{r}{a}\right)^2 - 16 \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{\delta_m}{k^2_{mn}} J_m(k_{mn} r/a) \cos m\theta \sin[k_{mn}(r/a)\cos\theta + \alpha a t - \tfrac{1}{2} m\pi] \sin\frac{\alpha a t}{2k_{mn}}. \tag{17}$$

The flow development is shown in figures 2–4 which display the summation of the preceding series at three typical times. The vorticity waves appear at the right and proceed to the left side of the container, from position E to W in figure 1. Since the frequency of each mode appearing in (17) decreases with increasing k_{mn} (i.e. with decreasing horizontal scale), for small times only the largest scale modes make a significant contribution to (17). As time increases, more and more waves become prominent until viscosity dissipates the motion. The viscous corrections can be shown to involve the multiplication of the right-hand sides of (16) or (17) by $\exp(-2E^{\frac{1}{2}}t)$. That is, the viscous spin-down time of each Rossby mode due to Ekman suction is *independent of scale*. A similar result was found in the case of a model rectangular β-plane ocean (Pedlosky 1965).

Preliminary experiments carried out at M.I.T. by R. Beardsley indicate reasonably good agreement with the theoretical predictions of phase speed and structure of the Rossby normal modes, and with the general nature of the spin-up dynamics.

As mentioned above, the Rossby modes found here in the sliced cylinder are completely similar to those found in the study of the simplest time-dependent bounded ocean models. The agreement of the theory presented here with experiments carried out in the sliced cylinder encourages the view that more complicated time-dependent problems of oceanographic interest can be profitably studied both theoretically and experimentally in this configuration. Indeed we may speculate that the process by which one oceanographic state of motion relaxes into another, as in a monsoonal change, may be qualitatively similar to the spin-down dynamics discussed here.

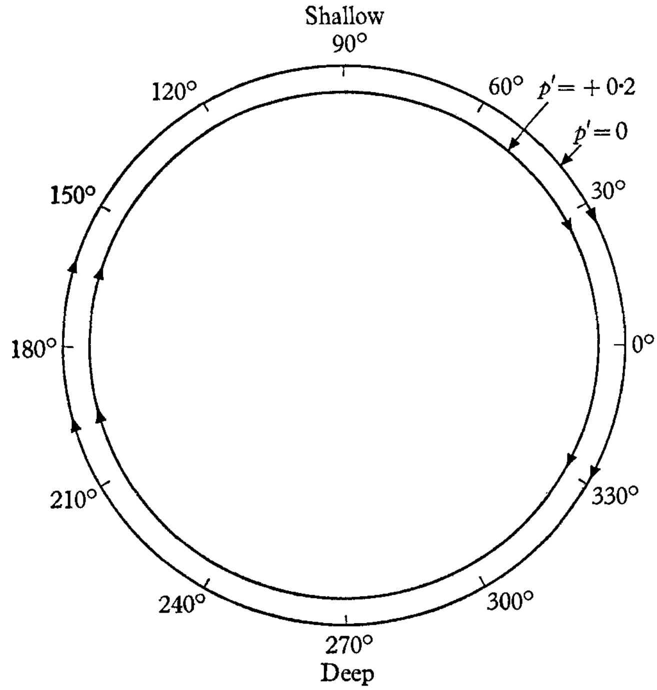

FIGURE 2. Pressure isolines or streamlines for rigid rotation in the sliced cylinder at time zero. The direction of the flow is shown; the shallowest region is due north, the deepest portion due south. In figures 2, 3 and 4, $p' = p/a^2$.

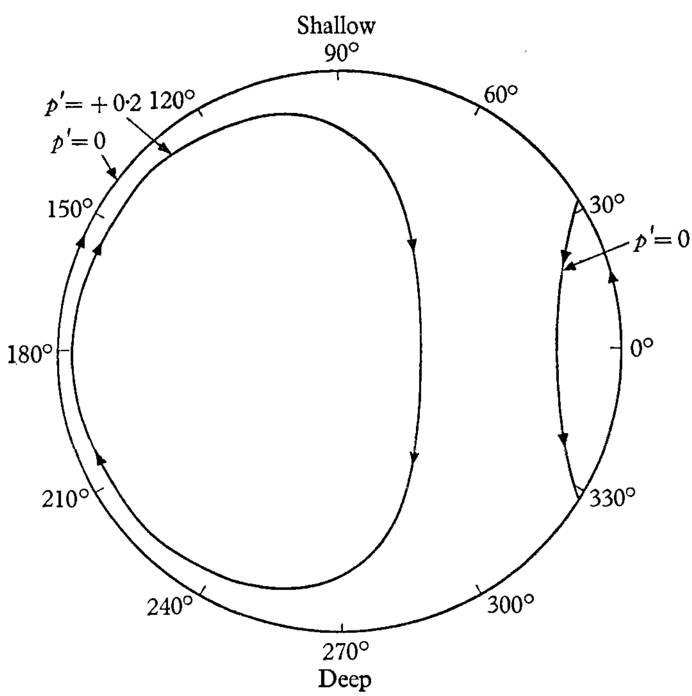

FIGURE 3. Pressure isolines at time $\alpha at = \pi$. A counter vortex appears in the east and begins to propagate west.

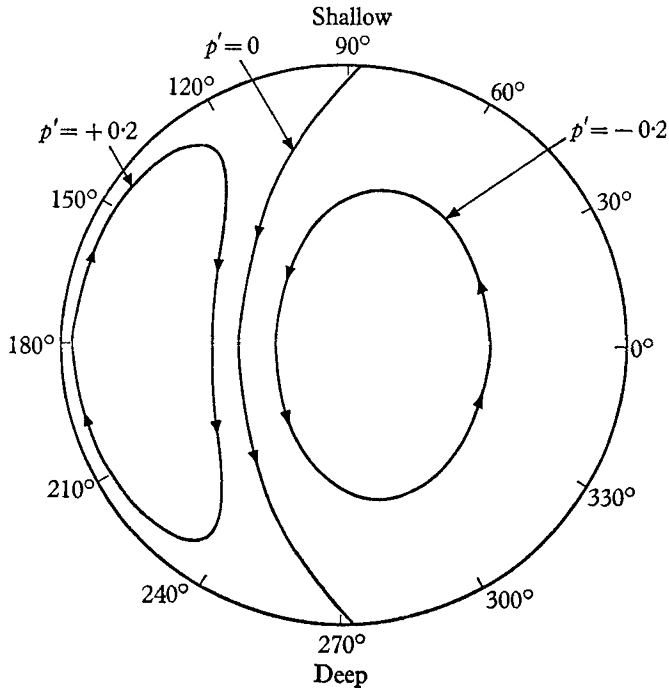

FIGURE 4. Streamlines at time $\alpha a t = 2\pi$.

4. Steady driven motions

As we noted, steady free geostrophic motion is not allowed in the sliced cylinder by the linear inviscid theory. Consider then, the following physical problem. Suppose the cylinder, rotating with an angular velocity Ω, has a top that is driving the fluid within the cylinder by rotating steadily at a slightly different rate. If α were zero, it is known that the fluid in the cylinder, aside from certain boundary-layer regions, would rotate as a solid body at the mean of the angular velocity of the container and its top. If α is different from zero but $\alpha \gg E^{\frac{1}{2}}$, a completely different flow pattern is established due to the constraint of the bottom slope which no longer allows an order one geostrophic flow. Order one flows will only appear in boundar-layer regions where the action of viscosity is sufficient to overcome the cited inviscid constraint.

The governing equations are again (1) while the boundary conditions are now, in non-dimensional form

$$\left.\begin{aligned} \mathbf{q} &= r\hat{\boldsymbol{\theta}} \quad \text{on} \quad z = 1, \\ \mathbf{q} &= 0 \quad \text{on} \quad r = a \quad \text{and on} \quad z = y\tan\alpha. \end{aligned}\right\} \tag{18}$$

The problem then decomposes in the following form. As previous ly discussed the velocity $\mathbf{q}_I$ in the interior of the cylinder is of necessity $O(E^{\frac{1}{2}})$, i.e.

$$\mathbf{q}_I = E^{\frac{1}{2}}\mathbf{q}_I^{(1)} + E\mathbf{q}_I^{(2)} + \ldots. \tag{19}$$

In order to satisfy (18) on $z = 1$, an Ekman layer of the standard type is required, so that in a region $(1-z) = O(E^{\frac{1}{2}})$ the inviscid dynamics gives way to Ekman boundary-layer flow. Matching the Ekman layer flow to the interior flow, $\mathbf{q}_I$, yields a condition on $\mathbf{q}_I^{(1)}$ at $z = 1$,

$$\mathbf{q}_I^{(1)}.\hat{\mathbf{k}} = 1 \quad \text{on} \quad z = 1. \tag{20}$$

That is, the suction into the Ekman layer by the inviscid flow must be such as to replace fluid in the Ekman layer which is flung out radially because of the faster rotation of the top. Similarly, at $z = y\tan\alpha$ another Ekman layer is required, but since the interior flow $\mathbf{q}_I$ is $O(E^{\frac{1}{2}})$ the Ekman layer suction is $O(E)$ (see Greenspan 1965), implying that the condition on $\mathbf{q}_I^{(1)}$ at $z = y\tan\alpha$ is

$$\mathbf{q}_I^{(1)}.\hat{\mathbf{k}} = \hat{\mathbf{j}}.\mathbf{q}_I^{(1)}\tan\alpha \quad \text{on} \quad z = y\tan\alpha. \tag{21}$$

Substitution of the expansion (19) into (1) yields the following interior equations for $\mathbf{q}_I^{(1)}$,

$$\left.\begin{aligned} 2\hat{\mathbf{k}}\times\mathbf{q}_I^{(1)} &= -\nabla p_I^{(1)}, \\ \nabla.\mathbf{q}_I^{(1)} &= 0, \end{aligned}\right\} \tag{22}$$

or, in terms of the pressure, $\mathbf{q}_I^{(1)}$ satisfies the relations

$$\mathbf{q}_I^{(1)} - (\mathbf{q}_I^{(1)}.\hat{\mathbf{k}})\,\hat{\mathbf{k}} = \tfrac{1}{2}\hat{\mathbf{k}}\times\nabla p_I^{(1)}, \quad \partial\mathbf{q}_I^{(1)}/\partial z = 0.$$

Thus, the interior motion is two-dimensional to this order, implying that

$$\mathbf{q}_I^{(1)}.\hat{\mathbf{k}}|_{z=1} = \mathbf{q}_I^{(1)}.\hat{\mathbf{k}}|_{z=0}, \tag{23}$$

or with (20) and (21)

$$\mathbf{q}_I.\hat{\mathbf{j}}\tan\alpha = 1. \tag{24}$$

The flux into the Ekman layer at the top is therefore balanced by a motion *across* lines of constant contour depth at a rate that makes the vertical velocities at the top and bottom of equal magnitude.

Writing (24) in terms of the pressure yields

$$\tan\alpha \quad \partial p_I^{(1)}/\partial x = 2. \tag{25}$$

Therefore $p_I^{(1)}$ is determined up to an arbitrary function of y, i.e.

$$p_I^{(1)} = \frac{2}{\tan\alpha}(x - \mathrm{X}(y)). \tag{26}$$

The determination of the function $\mathrm{X}(y)$ which yields the flow along the lines of constant height must await a discussion of the process by which the boundary condition (18) on $r = a$ is satisfied. The simple interior flow cannot itself satisfy the conditions on the side wall and a side wall boundary layer of thickness $E^{\frac{1}{3}}$ is required. A boundary layer of thickness $E^{\frac{1}{4}}$ does not arise, to this degree of approximation, because the tangential component of the interior velocity at the side wall differs from the wall velocity by only $O(E^{\frac{1}{2}})$. An order one discontinuity would be necessary to require the thicker layer.

Let the total velocity and pressure away from the Ekman layers at $z = 1$ and $y\tan\alpha$ be written in the form

$$\left.\begin{aligned} \mathbf{q} &= E^{\frac{1}{2}}\mathbf{q}_I^{(1)}(x,y) + E\mathbf{q}_I^{(2)}(x,y,z) + \ldots + \mathbf{q}_b(\eta,\theta,z) + \ldots, \\ p &= E^{\frac{1}{2}}p_I^{(1)}(x,y) + Ep_I^{(2)}(x,y,z) + \ldots + E^{\frac{1}{2}}p_b(\eta,\theta,z) + \ldots, \end{aligned}\right\} \tag{27}$$

where $$\eta = (a-r)E^{-\frac{1}{3}}$$
and $$\hat{\mathbf{k}}.\mathbf{q}_b = E^{\frac{1}{3}}W, \quad \hat{\boldsymbol{\theta}}.\mathbf{q}_b = E^{\frac{1}{3}}V, \quad \hat{\mathbf{r}}.\mathbf{q}_b = E^{\frac{1}{3}}U.$$

The perturbation boundary-layer functions $\mathbf{q}_b, p_b = P$, must decay exponentially fast as $\eta \to \infty$ so that in the interior $\mathbf{q} = \mathbf{q}_I$.

Substituting the expansion (27) into (1) and then taking the limit $E \to 0$, fixed η, yields the boundary-layer equations

$$\left.\begin{aligned} P_\eta + 2V &= 0, \\ a^{-1}P_\theta + 2U - V_{\eta\eta} &= 0, \\ P_z - W_{\eta\eta} &= 0, \\ W_z - U_\eta + a^{-1}V_\theta &= 0. \end{aligned}\right\} \tag{28}$$

The velocity variables may be eliminated in favour of the pressure yielding

$$P_{\eta\eta\eta\eta\eta\eta} - 4P_{zz} = 0, \tag{29}$$

where
$$\left.\begin{aligned} V &= -\tfrac{1}{2}P_\eta, \\ U &= -\tfrac{1}{2}a^{-1}P_\theta - \tfrac{1}{4}P_{\eta\eta\eta}, \\ W_{\eta\eta} &= P_z. \end{aligned}\right\} \tag{30}$$

The boundary conditions for P are determined from the corresponding conditions on the individual velocity components as follows. There are Ekman layers at $z = 1$ and $z = y\tan\alpha$ within the $E^{\frac{1}{3}}$ layer. It can easily be shown that the Ekman suction velocity into these boundary layers is $O(E^{\frac{1}{2}})$ so that

$$\left.\begin{aligned} W &= 0 \quad \text{on} \quad z = 1, \\ W &= V\cos\theta \ \tan\alpha \quad \text{on} \quad z = y\tan\alpha, \end{aligned}\right\} \tag{31}$$

which imply that
$$\left.\begin{aligned} P_z &= 0 \quad \text{on} \quad z = 1, \\ P_z &= -\tfrac{1}{2}\tan\alpha \ \ \cos\theta \ \ P_{\eta\eta\eta} \quad \text{on} \quad z = y\tan\alpha. \end{aligned}\right\} \tag{32}$$

In addition, to match the interior velocity to the boundary conditions on $r = a$

$$\left.\begin{aligned} &P_\eta = 0 \\ &\tfrac{1}{2}a^{-1}P_0 + \tfrac{1}{4}P_{\eta\eta\eta} = u_I(a,\theta) = \alpha^{-1}(\mathrm{X}'(a\sin\theta)\cos\theta + \sin\theta), \\ &P_{\eta\eta\eta\eta} = 0 \end{aligned}\right\} \tag{33}$$

and to ensure a boundary-layer behaviour

$$P \to 0 \quad \text{as} \quad \eta \to \infty.$$

The solution to (29) is of the form

$$P = \sum_n A_n e^{-\lambda_n \eta} \cos[\tfrac{1}{2}\lambda_n^3(1-z)]. \tag{34}$$

The boundary conditions (32) lead to the eigenvalue equation for λ_n

$$\tan[\tfrac{1}{2}\lambda_n^3(1 - a\tan\alpha\sin\theta)] = \tan\alpha\cos\theta. \tag{35}$$

Only roots of (35) with positive real part for λ_n are acceptable. The remaining discussion, for simplicity, will be concerned with small α (but $\alpha \gg E^{\frac{1}{2}}$). The qualitative nature of the solution is not affected for α of $O(1)$.

For small α, one root of (35) is approximately

$$\lambda_0^3 = 2\alpha\cos\theta, \tag{36}$$

while the higher roots are given to order α by

$$\lambda_n^3 = \pm 2n\pi \quad (n = 1, 2, 3, \ldots). \tag{37}$$

The general solution for the pressure is then

$$\begin{aligned} P = {} & A_{01}\exp\left(-|2\alpha\cos\theta|^{\frac{1}{3}}\eta\right)\left(\frac{\cos\theta+|\cos\theta|}{2\cos\theta}\right) \\ & + \{\tfrac{1}{2}A_{02}(1-\sqrt{3}i)\exp\left(-\tfrac{1}{2}|2\alpha\cos\theta|^{\frac{1}{3}}(1+\sqrt{3}i)\eta\right) \\ & + \tfrac{1}{2}A_{03}(1+\sqrt{3}i)\exp\left(-\tfrac{1}{2}|2\alpha\cos\theta|^{\frac{1}{3}}(1-\sqrt{3}i)\eta\right)\}\left(\frac{\cos\theta-|\cos\theta|}{2\cos\theta}\right) \\ & + \sum_{n=1}^{\infty}\cos n\pi(z-1)\{A_{n1}\exp\left(-(2n\pi)^{\frac{1}{3}}\eta\right) \\ & + \tfrac{1}{2}A_{n2}(1-\sqrt{3}i)\exp\left(-\tfrac{1}{2}(2n\pi)^{\frac{1}{3}}(1+\sqrt{3}i)\eta\right) \\ & + \tfrac{1}{2}A_{n3}(1+\sqrt{3}i)\exp\left(-\tfrac{1}{2}(2n\pi)^{\frac{1}{3}}(1-\sqrt{3})i)\eta\right)\}. \end{aligned} \tag{38}$$

The boundary conditions (33) on $r = a$ lead to the following relations

$$\left.\begin{aligned} A_{01} = 0, \quad A_{02}+A_{03} = 0, \\ A_{n1} = 0, \quad A_{n2}+A_{n3} = 0, \end{aligned}\right\} \tag{39}$$

and

$$\left\{\frac{d}{d\theta}A_{02}+2a\alpha|\cos\theta|\,A_{02}\right\}\left(\frac{\cos\theta-|\cos\theta|}{2\cos\theta}\right) = \frac{2ia}{\alpha\sqrt{3}}\left(\mathrm{X}'(a\sin\theta)\cos\theta+\sin\theta\right), \tag{40}$$

$$dA_{n2}/d\theta + n\pi a A_{n2} = 0. \tag{41}$$

From the first of these, it is evident that the boundary layer occurs only on part of the side wall for which $\cos\theta < 0$, the western side of the basin. Furthermore, the form of (40) implies that the right-hand side of this equation must be identically zero when $\cos\theta > 0$. This then, allows us to determine the unknown function $\mathrm{X}(y)$ and to complete the interior solution:

$$\mathrm{X}(y) = (a^2-y^2)^{\frac{1}{2}}, \tag{42}$$

$$p_I = 2\alpha^{-1}E^{\frac{1}{2}}(x-(a^2-y^2)^{\frac{1}{2}}). \tag{43}$$

Finally, the coefficient function A_{02} can be evaluated by integrating (40) and, to terms of $O(\alpha)$, the result is

$$A_{02} = -(4ai/\alpha\sqrt{3})\cos\theta + iK_{02}, \tag{44}$$

where K_{02} is an arbitrary constant. To ensure that the mass flux in the boundary layer balances the flux in the interior (there is no net transport at either end of the boundary layer $\theta = \pm\frac{1}{2}\pi$) implies that K_{02} is zero.

The solution of (41) is

$$A_{n2} = K_{n2}e^{-n\pi a\theta}, \tag{45}$$

but symmetry and periodicity requirements lead at once to the conclusion that all constants K_{n2} are zero. Thus only A_{02}, A_{03} are non-trivial functions.

The complete expression for the pressure is

$$p = 2E^{\frac{1}{2}}\alpha^{-1}\left[(x-(a^2-y^2)^{\frac{1}{2}}) + \frac{8a\cos\theta}{\sqrt{3}}\left\{\frac{\cos\theta-|\cos\theta|}{2\cos\theta}\right\}\operatorname{Im}\exp\left(-\tfrac{1}{2}|2\alpha\cos\theta|^{\frac{1}{3}}(1+\sqrt{3}\,i)\,\eta-\tfrac{1}{3}\pi i\right)\right]. \quad (46)$$

To this order in α, the pressure serves as a stream function for the horizontal velocity and the flow is shown in figure 5.

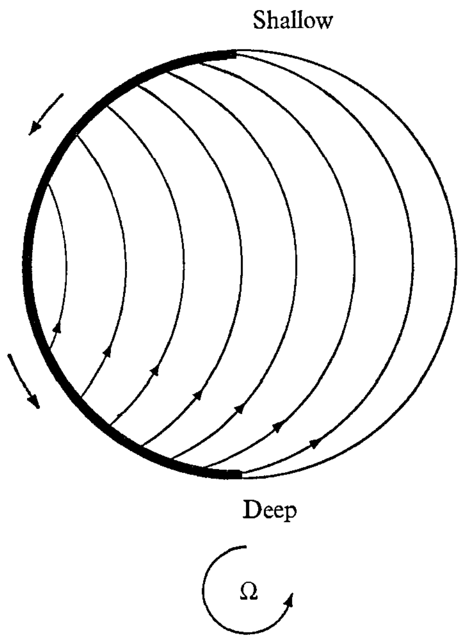

FIGURE 5. Streamlines of the steady driven flow, with the boundary layer shown on the eastern side of the cylinder.

A very weak boundary layer on the eastern side of the cylinder, $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$, is required to bring the $O(E^{\frac{1}{2}})$ tangential flow to zero on $r = a$. However, this is not a significant part of the motion, especially in comparison to the western boundary layer, and will not be discussed further.

The constraint due to the bottom incline has completely altered the character of the symmetric circulation described earlier in the case $\alpha = 0$. The only $O(1)$ flow occurs in the upper Ekman layer. In the main body of the fluid, a very slow drift of order $E^{\frac{1}{2}}$ is predicted along with the presence of a relatively strong boundary layer on *only* one side of the cylinder of thickness $E^{\frac{1}{3}}$ wherein the velocities are $O(E^{\frac{1}{6}})$. Thus, the removal of the free geostrophic mode manifests itself in the appearance of a non-axially symmetric side-wall boundary layer, or as the appearance of Rossby modes in the time-dependent problem. In both cases a unique direction is picked out by the rotation vector and the bottom slope.

The similarity of this solution of a laboratory model with certain solutions (e.g. Munk & Carrier 1950) of wind-driven ocean circulation *model* problems is striking. It is natural to identify the boundary current with the Gulf Stream model produced by the aforementioned β-plane viscous theory. This again encourages us to believe that more complicated problems relevant to the understanding of the wind-driven ocean circulation can be further studied both experimentally and theoretically in this configuration.

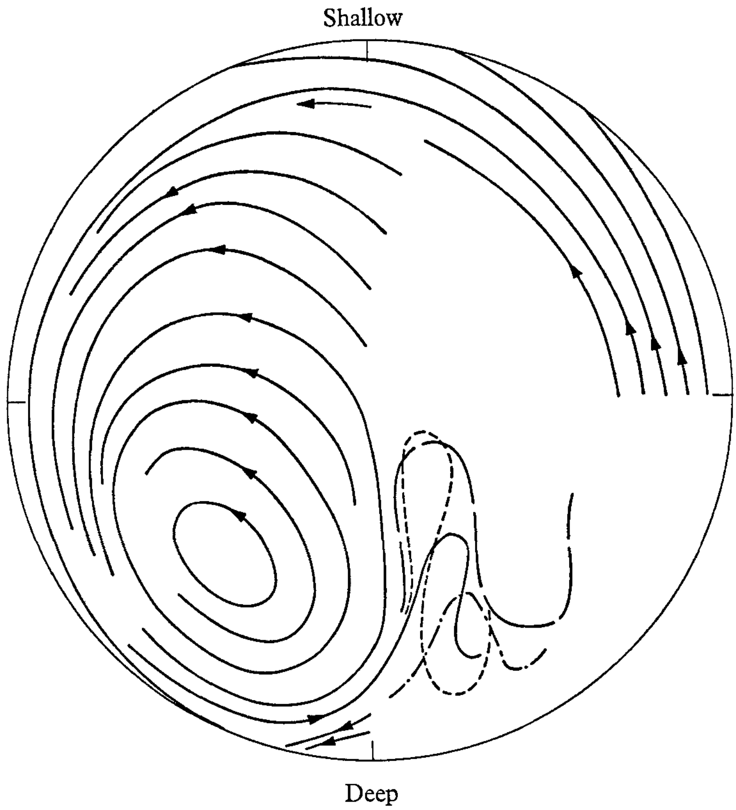

FIGURE 6. A sketch of the stream lines of the observed steady flow, with the observed meanders apparent in the south-eastern portion of the cylinder (by courtesy of R. Beardsley).

Mr R. Beardsley has experimentally investigated the problem analysed here and a photograph and schematic drawing of what he observed appears in figures 6 and 7 (plate 1). The experiments verify the existence and position of the boundary current phenomena as well as the two-dimensional character of the flow for small α. There is, however, a dramatic divergence of the experimental result from the theory. Emanating from the deepest portion of the cylinder ($r = a, \theta = \frac{3}{2}\pi$) (where the $E^{\frac{1}{4}}$ layer formally becomes infinitely thick and where the interior solution has a branch point singularity) are quasi-steady meanders which seem to occupy only the 'eastern' section of the cylinder. We are not yet in a position to explain the presence of the meanders, but the similarity of these laboratory meanders with those found in the real ocean is another striking demonstration of the geophysical relevance of this laboratory configuration.

Finally, it should be noted that many similar problems can be formulated in this configuration which are of geophysical relevance. The problem where the cylinder top drives the fluid in an unsteady manner combines some aspects of both problems discussed here. Certainly the corresponding non-linear problems would be of great interest. Moreover, the introduction of density stratification, caused by either controlled salinity or temperature distributions would undoubtedly produce many significant new effects. Many of these problems are being studied at present.

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FIGURE 7. A photograph showing the observed meanders marked by dye (by courtesy of R. Beardsley).